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Surface-breaking crack in an elastic half-space

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Abstract. A semi-infinite crack terminating at the boundary of an elastic half-space is considered. It is assumed that the crack is subjected to a mode-I load applied in a finite region remote from the crack plane, and the boundary of the half-space and the crack surfaces are free of tractions. The problem is formulated in terms of a hypersingular integral equation with respect to the relative crack-face separation defined over the region occupied by the crack. The behaviour of the solution near the corner point where the crack edge intersects the boundary is analysed and results which show the dependence of the stress singularity exponent on the angle of inclination of the crack edge are presented.

Key words: surface-breaking crack, elastic half-space, singularity exponent, termination point, crack-opening displacement, hypersingular integral equation.

1. Introduction

The problem to be addressed in this work may be explained by reference to Figures 1 and 2. Figure 1 illustrates a surface-breaking crack. It is idealised as a plane surface S which intersects with a specimen boundary, idealised in the figure as the plane $x_3 = 0$. Away from the specimen boundary, if linear elasticity is assumed, the stresses display the usual square-root singular behaviour in the vicinity of the crack edge. It is of fundamental interest to comprehend the singular behaviour of the stresses in the vicinity of a point (O in Figure 1) of intersection of the crack boundary with the surface of the specimen. This can be investigated by magnifying the vicinity of the point O as illustrated in Figure 2. Here, the crack appears as an infinite plane region whose edge, now straight, intersects the free surface at O. Relative to polar coordinates (ρ, θ, φ), the displacement field is assumed to have the asymptotic form

$$\boldsymbol{u} = \rho^{\Lambda}\mathbf{U}(\Lambda; \theta, \varphi). \tag{1}$$

The displacement field $\boldsymbol{u}$ has to satisfy the Lamé equations of linear elasticity and the conditions that traction components are zero on the crack faces and on the specimen surface $x_3 = 0$.

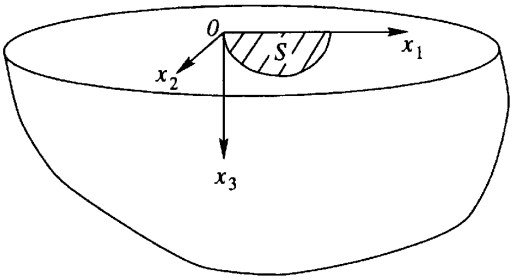

Figure 1. Surface-breaking crack in a half-space.

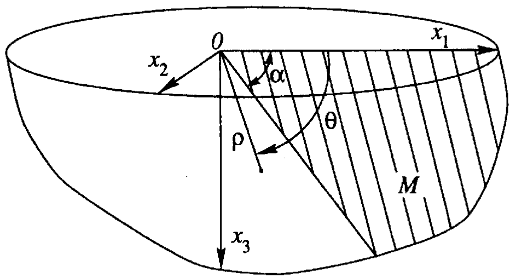

Figure 2. Crack geometry near the termination point O.

The corresponding stresses are proportional to $\rho^{\Lambda-1}$. Interest in singular stresses restricts concern to values of Λ for which $\mathrm{Re}\Lambda < 1$. Boundedness of displacements corresponds to $\mathrm{Re}\Lambda \geqslant 0$. The weaker requirement of locally integrable energy density provides the restriction $\mathrm{Re}\Lambda > -\frac{1}{2}$. In this work, real values of Λ in the range $-\frac{1}{2} < \Lambda < 1$ will be investigated. The problem already has a substantial history. Benthem [1, 2] took the crack and the straight edge to be normal to the free surface so that, relative to Figure 2, it occupies the quarter-plane $\{x_1 \geqslant 0, x_2 = 0, x_3 \geqslant 0\}$. He found three independent solutions of the form (1). One, corresponding to pure mode-I symmetry (only the displacement u_2 is discontinuous across S), displayed a smaller value of Λ than the other two (for which only u_2 had no discontinuity corresponding to mixtures of modes II and III). All Λ's found by Benthem were real and positive. The values of Λ were found to depend on Poisson's ratio ν. Folias [3, 4] considered the mode-I problem for the same geometry but obtained real and negative values for Λ[1]. The work of Benthem and of Folias was semi-analytic, employing eigenfunction expansions whose coefficients satisfied an infinite system of linear algebraic equations. Bažant and Estenssoro [5] performed numerical analysis by employing a finite-element formulation. The flexibility thereby afforded allowed them to address the problem of a plane crack with any orientation relative to the free surface. When specialized to the geometry considered by Benthem, their results tended to agree with Benthem (definitely not Folias), mesh refinement producing results whose formal extrapolations were close to those of Benthem. Accordingly, all of their presented results were obtained by extrapolation from their computed results, rather than directly.

There appears still to be some confusion about the correct value of Λ. The analytical solution derived by Leung and Su [6] for the same geometry as in [1] by superposition of two singular solutions – the one for a crack in an elastic space and the other one due to a singular tractions on the free surface – displays no dependence of Λ on the Poisson ratio ν. Dhondt [7] constructed the Green function for a mode-I semi-circular crack and deduced that the corresponding stress intensity factor has a logarithmic singularity at the free surface. The results of Leung and Su [6] and Dhondt [7] disagree with those of Benthem [1], Bažant and Estenssoro [5], and Folias [3, 4].

Therefore, study of the problem by a different method seems justified. Any problem for a crack in a body, whether surface-breaking or not, can be formulated quite concisely as a system of hypersingular integral equations for the jump in displacements across the crack surface S. The implementation is particularly easy for a crack in an infinite body [8, 9], but

[1] This motivated our decision to investigate the range $-\frac{1}{2} < \Lambda < 1$ in the present work.

it can also be carried through in the case of a crack in a half-space, because the associated Green's function is known explicitly [10]. For mode-I loading of a crack occupying a region on the plane $x_2 = 0$, the relevant equation has been derived and discussed by Martin *et al.* [8]. However numerical analysis of the corner singularity was not attempted. The present study fills that gap. Thus, for the asymptotic problem, the crack is taken to occupy the portion

$$M = \{x_1 = \rho\cos\theta, x_3 = \rho\sin\theta, 0 < \theta < \alpha, 0 < \rho < \infty\} \tag{2}$$

of the plane $x_2 = 0$ (equivalently, $\varphi = 0$ or π). The equation governing the jump in the displacement component u_2 is given explicitly in the next section, and some of its essential properties are investigated in Section 3. Section 4 reports results, which were obtained by discretization and investigation of the determinant of the resulting matrix as a function of Λ, for different values of Poisson's ratio ν and crack intersection angle α. The governing equations are homogeneous, so the allowed value of Λ is that for which the determinant is zero. Full discussion is deferred until after presentation of the results, but it is perhaps appropriate to state here that only one real Λ was found for each ν and α, and these agree closely with the values found by Benthem when $\alpha = \pi/2$. The integral-equation method can also be applied to the case of a surface-breaking crack extending dynamically. Such a study is in progress and will be reported separately.

2. Governing equations

Consider an isotropic elastic half-space $\mathbf{R}^3_+ = \{\boldsymbol{x} \in \mathbf{R}^3 : x_3 > 0\}$ containing a crack in the x_1x_3-plane. Formally, the crack occupies the region $\{\boldsymbol{x} : (x_1, x_3) \in M,\ x_2 = 0\}$. In the general case, M may or may not contain points of the boundary $x_3 = 0$ of the half-space – that is, the crack may or may not be surface-breaking.

Assume that the crack is subjected to a mode-I symmetric load applied in a finite region V remote from the crack plane, and the boundary of the half-space $\partial\mathbf{R}^3_+$ and the crack surfaces $\Gamma_+,\ \Gamma_-$ are free of tractions.

The displacement field in the body consists of two parts: the displacement field, $\boldsymbol{u}^0$ say, which would be generated by the applied load in the absence of the crack, and the displacement field, $\boldsymbol{u}$ say, induced by the presence of the crack.

A formal application of Betti's theorem yields

$$u_p(\boldsymbol{x}') = \int_M [u_i](\boldsymbol{x})\, c_{ijkl} \frac{\partial G_k^{(p)}}{\partial x_l}(\boldsymbol{x}, \boldsymbol{x}')\, n_j(\boldsymbol{x})\, \mathrm{d}s_x, \tag{3}$$

where $[\boldsymbol{u}] = ([u_1], [u_2], [u_3])$ denotes the relative displacement of the crack faces, c_{ijkl} are the components of the tensor of elastic moduli,

$$c_{ijkl} = \frac{E}{2(1+\nu)} \left\{ \frac{2\nu}{1-2\nu}\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} \right\},$$

E and ν are the Young modulus and the Poisson ratio respectively, and $\mathbf{G}(\boldsymbol{x}, \boldsymbol{x}') = (G^{(1)}(\boldsymbol{x}, \boldsymbol{x}'), G^{(2)}(\boldsymbol{x}, \boldsymbol{x}'), G^{(3)}(\boldsymbol{x}, \boldsymbol{x}'))$ is the Green tensor for the half-space: $G_k^{(i)}(\boldsymbol{x}, \boldsymbol{x}')$ is the k-component of the displacement at $\boldsymbol{x}$ produced by a unit body force applied in the i-direction at $\boldsymbol{x}'$. Since the system of coordinates used here is the same as in [10], we do not present the formulae for the components of the Green tensor $\mathbf{G}(\boldsymbol{x}, \boldsymbol{x}')$ and refer the reader to [10][2].

[2] The components of $G(\boldsymbol{x}, \boldsymbol{x}')$ are also given explicitly in [8], except that [8] discusses the half-space $x_2 > 0$, resulting in interchange of suffixes 2 and 3 relative to our work.

The corresponding stress field is given by

$$\sigma_{rs}(\boldsymbol{u}; \boldsymbol{x}') = c_{rspq} \int_M [u_i](\boldsymbol{x})\, c_{ijkl} \frac{\partial^2 G_k^{(p)}(\boldsymbol{x}, \boldsymbol{x}')}{\partial x_l \partial x_q'}\, n_j(\boldsymbol{x})\, \mathrm{d}s_x. \tag{4}$$

For mode-I loading, only the second component $[u_2]$ of the displacement field $[\boldsymbol{u}]$ differs from zero; it vanishes like the square root of $\alpha - \theta$ as $\theta \to \alpha$, with ρ fixed, and satisfies the equation

$$\int_M [u_2](x_1, x_3)\, (K^*(x_1 - x_1', x_3 - x_3') + K_0(x_1, x_3; x_1', x_3'))\, \mathrm{d}x_1\, \mathrm{d}x_3$$
$$= \sigma_{22}(\boldsymbol{u}^0; x_1', x_3'), \;\; (x_1', x_3') \in M. \tag{5}$$

The kernel $K(x_1, x_3; x_1', x_3') = K^*(x_1 - x_1', x_3 - x_3') + K_0(x_1, x_3; x_1', x_3')$ is hypersingular. Its singular part K^* is associated with the Green tensor for a whole space and has the form

$$K^*(x_1 - x_1', x_3 - x_3') = -\frac{E}{8\pi(1-\nu^2)R_1^3}. \tag{6}$$

The regular part K_0 is associated with the 'correction' required to render the boundary $x_3 = 0$ free of tractions, and is given by

$$K_0(x_1, x_3; x_1', x_3')$$
$$= -\frac{E}{8\pi(1-\nu^2)} \left[\frac{2\nu - 5}{R_2^3} + \frac{18 x_3 x_3'}{R_2^5} + \frac{12(1-\nu)(1-2\nu)}{R_2(R_2 + x_3 + x_3')^2} + \frac{6\nu(x_3 + x_3')^2}{R_2^5} + \frac{6\nu(3-4\nu)(x_1 - x_1')^2}{R_2^5} \right], \tag{7}$$

where

$$R_1 = \sqrt{(x_1 - x_1')^2 + (x_3 - x_3')^2}, \qquad R_2 = \sqrt{(x_1 - x_1')^2 + (x_3 + x_3')^2}.$$

Equation (5) follows from (4) and the boundary condition

$$\sigma_{22}(\boldsymbol{u}^0; \boldsymbol{x}') + \sigma_{22}(\boldsymbol{u}; \boldsymbol{x}') = 0, \quad \boldsymbol{x}' \in \Gamma_\pm.$$

It coincides (apart from replacement of x_1 with x and x_3 with y) with the equation presented and analysed by Martin *et al.* [8]. They discussed existence and uniqueness of its solution and remarked on the need for quantitative evaluation of the singularity at a point where the crack edge meets the free surface $x_3 = 0$. The present study has the latter objective, for which it suffices to specialize the domain M to the form (2).

Numerical calculations require explicit treatment of the hypersingular part K^* of the kernel in (5). The integral is interpreted in the finite-part Hadamard sense [11], or equivalently in the sense of generalized functions [12]. Movchan and Willis [9] proposed the addition and subtraction of the first two terms of the Taylor expansion of $[u_2]$ about the point of evaluation, coupled with explicit treatment of the 'linear approximation' to $[u_2]$ by transformation to an integral around the boundary ∂M, exploiting the fact that, in two dimensions, $\Delta(1/r) = 1/r^3$, where $r^2 = (x_1 - x_1')^2 + (x_3 - x_3')^2$ and Δ is the Laplacian. Adapting that proposal to the present situation, we may define the singular integral by

$$\begin{aligned}
&\int_M [u_2](x_1, x_3) K^*(x_1 - x_1', x_3 - x_3')\,\mathrm{d}x_1\,\mathrm{d}x_3 \\
&= \int_{M_R} K^*(x_1 - x_1', x_3 - x_3') \left\{ [u_2](x_1, x_3) - [u_2](x_1', x_3') - (x_1 - x_1') \frac{\partial [u_2]}{\partial x_1}(x_1', x_3') \right. \\
&\qquad\qquad \left. - (x_3 - x_3') \frac{\partial [u_2]}{\partial x_3}(x_1', x_3') \right\} \mathrm{d}x_1\,\mathrm{d}x_3 \\
&\quad + \int_{\partial M_R} \mathrm{d}s \left[\frac{\partial \Theta}{\partial n} \left\{ [u_2](x_1', x_3') + (x_1 - x_1') \frac{\partial [u_2]}{\partial x_1}(x_1', x_3') \right. \right. \\
&\qquad\qquad \left. + (x_3 - x_3') \frac{\partial [u_2]}{\partial x_3}(x_1', x_3') \right\} \\
&\qquad\qquad \left. - \Theta \left\{ n_1 \frac{\partial [u_2]}{\partial x_1}(x_1', x_3') + n_3 \frac{\partial [u_2]}{\partial x_3}(x_1', x_3') \right\} \right] \\
&\quad - \frac{E}{8\pi(1 - \nu^2)} \int_{M \setminus M_R} \frac{[u_2](x_1, x_3)\,\mathrm{d}x_1\,\mathrm{d}x_3}{((x_1 - x_1')^2 + (x_3 - x_3')^2)^{3/2}},
\end{aligned} \tag{8}$$

where $M_R = \{\boldsymbol{x} \in M : x_1^2 + x_3^2 < R^2\}$, and $\Theta = -E[8\pi(1 - \nu^2) R_1]^{-1}$. It is only necessary that $R > \rho' = (x_1'^2 + x_3'^2)^{1/2}$, so that the last integral in (8) is regular. In the computations, in fact, R was taken to be $2\rho'$.

3. Investigation of the singularity

We study the singularity by postulating a solution of the *homogeneous* integral equation (5) of the form

$$[u_2] = \rho^{\Lambda} f(\theta). \tag{9}$$

Thus, $\boldsymbol{u}^0 = 0$ and $\sigma_{22}(\boldsymbol{u}^0; x_1, x_3) = 0$, and a non-trivial solution of the form (9) will exist only when Λ is an eigenvalue. The integrals with respect to ρ are either treated explicitly or reduced to integrals over finite intervals and evaluated numerically (see Appendix A). In turn, the terms have the forms listed below.

$$\begin{aligned}
&\int_{M_R} K^*(x_1 - x_1', x_3 - x_3') \{ [u_2](x_1, x_3) - [u_2](x_1', x_3') \\
&\qquad - (x_1 - x_1')[u_2]_{,1}(x_1', x_3') - (x_3 - x_3')[u_2]_{,3}(x_1', x_3') \}\,\mathrm{d}x_1\,\mathrm{d}x_3 \\
&= -\frac{E(\rho')^{\Lambda - 1}}{8\pi(1 - \nu^2)} \int_0^{\alpha} \{ I_1(\theta, \theta') f(\theta) - I_2(\theta, \theta') f(\theta') - I_3(\theta, \theta') f'(\theta') \}\,\mathrm{d}\theta,
\end{aligned} \tag{10}$$

$$\int_0^{2\rho'} \mathrm{d}\rho \left[\frac{1}{\rho} \frac{\partial \Theta}{\partial \theta} \{[u_2](x_1', x_3') + (x_1 - x_1')[u_2]_{,1}(x_1', x_3') + (x_3 - x_3')[u_2]_{,3}(x_1', x_3')\} \right.$$
$$\left. - \frac{\Theta}{\rho} \frac{\partial}{\partial \theta} \{(x_1 - x_1')[u_2]_{,1}(x_1', x_3') + (x_3 - x_3')[u_2]_{,3}(x_1', x_3')\} \right]_{\theta=0}^{\alpha}$$
$$= -\frac{E(\rho')^{\Lambda-1}}{8\pi(1-\nu^2)} \{[J_1(\alpha, \theta') - J_1(0, \theta')] f(\theta')$$
$$+ [J_2(\alpha, \theta') - J_2(0, \theta')] f'(\theta')\}, \tag{11}$$

$$\int_0^{\alpha} (2\rho')\, \mathrm{d}\theta \left[\frac{\partial \Theta}{\partial \rho} \{[u_2](x_1', x_3') + (x_1 - x_1')[u_2]_{,1}(x_1', x_3') + (x_3 - x_3')[u_2]_{,3}(x_1', x_3')\} \right.$$
$$\left. - \Theta \frac{\partial}{\partial \rho} \{(x_1 - x_1')[u_2]_{,1}(x_1', x_3') + (x_3 - x_3')[u_2]_{,3}(x_1', x_3')\} \right]_{\rho=2\rho'}$$
$$= -\frac{E(\rho')^{\Lambda-1}}{8\pi(1-\nu^2)} \{J_3(\theta') f(\theta') + J_4(\theta') f'(\theta')\}, \tag{12}$$

$$\int_0^{\alpha} \mathrm{d}\theta \int_{2\rho'}^{\infty} \rho\, \mathrm{d}\rho K^*(x_1 - x_1', x_3 - x_3')[u_2](x_1, x_3)$$
$$= -\frac{E(\rho')^{\Lambda-1}}{8\pi(1-\nu^2)} \int_0^{\alpha} \mathrm{d}\theta I_4(\theta, \theta') f(\theta), \tag{13}$$

$$\int_M K_0(x_1, x_3; x_1', x_3')[u_2](x_1, x_3)\, \mathrm{d}x_1\, \mathrm{d}x_3 = (\rho')^{\Lambda-1} \int_0^{\alpha} \mathrm{d}\theta \tilde{K}_0(\Lambda; \theta, \theta') f(\theta). \tag{14}$$

The functions I_1 to I_4, J_1 to J_4 and $\tilde{K}_0$ are given explicitly in Appendix A. The homogeneous equation (5) now implies that

$$\int_0^{\alpha} \{I_1(\theta, \theta') f(\theta) - I_2(\theta, \theta') f(\theta') - I_3(\theta, \theta') f'(\theta')\}\, \mathrm{d}\theta$$
$$+ \int_0^{\alpha} I_4(\theta, \theta') f(\theta) \mathrm{d}\theta - \frac{8\pi(1-\nu^2)}{E} \int_0^{\alpha} \tilde{K}_0(\Lambda; \theta, \theta') f(\theta)\, \mathrm{d}\theta$$
$$+ [J_1(\alpha, \theta') - J_1(0, \theta') + J_3(\theta')] f(\theta') + [J_2(\alpha, \theta') - J_2(0, \theta') + J_4(\theta')] f'(\theta') = 0. \tag{15}$$

It is appropriate to discuss the anticipated form of $f(\theta)$. The regularization procedure ensures that all integrals exist for any $\theta' \in (0, \alpha)$. The expression on the left side of (15) could, however, become singular as $\theta' \to 0$ or $\theta' \to \alpha$. Considering first $\theta' \to \alpha$, we may verify that

$$J_1(\alpha, \theta') \sim -2(\alpha - \theta')^{-1} \quad \text{as } \theta' \to \alpha. \tag{16}$$

Therefore, the left side of (15) has an unacceptably strong singularity, unless $f(\alpha) = 0$. This is nothing more than the requirement that the relative displacement should tend to zero as the crack edge is approached it will be assumed without further discussion that $f(\theta)$ has the form $(\alpha-\theta)^{1/2}g(\theta)$, where $g(\theta)$ is bounded near $\theta=\alpha$, consistent with the known behaviour of the relative displacement near any smooth crack boundary. It can be verified similarly that

$$J_1(0,\theta') \sim 2(\theta')^{-1} \quad \text{as } \theta' \to 0. \tag{17}$$

There is, however, a compensating term that comes from the integral involving the 'image' term $\tilde{K}_0$. In fact, asymptotically, for $\theta, \theta' \to 0$,

$$\tilde{K}_0(\Lambda;\theta,\theta') \sim \frac{E}{4\pi(1-\nu^2)}\left\{\frac{1}{(\theta+\theta')^2} - \frac{12\theta\theta'}{(\theta+\theta')^4}\right\}, \tag{18}$$

which is consistent precisely with the 'image' term for the corresponding kernel for a crack in two dimensions [8]

$$\int_{-\infty}^{\infty} K_0(x_1,x_3;x_1',x_3')\,\mathrm{d}x_1 = \frac{E}{4\pi(1-\nu^2)}\left\{\frac{1}{(x_3+x_3')^2} - \frac{12x_3x_3'}{(x_3+x_3')^4}\right\}. \tag{19}$$

Asymptotically, therefore, as $\theta' \to 0$,

$$\frac{4\pi(1-\nu^2)}{E}\int_0^{\alpha} \tilde{K}_0(\Lambda;\theta,\theta')f(\theta)\,\mathrm{d}\theta$$
$$\sim f(0)\int_0^{\alpha}\left\{\frac{1}{(\theta+\theta')^2} - \frac{12\theta\theta'}{(\theta+\theta')^4}\right\}\mathrm{d}\theta = -\frac{f(0)}{\theta'}. \tag{20}$$

Thus, as expected, the opening of the crack at its intersection with the free surface is allowed to be finite and non-zero.

The procedure now is to discretize (15), and to calculate the determinant of the resulting matrix (A, say) multiplying the vector of nodal values of $f(\theta')$ as a function of Λ. An approximation to the desired eigenvalue is then delivered by a value of Λ for which the determinant vanishes.

The interval $(0,\alpha)$ was sub-divided into N equal intervals $\{(\theta_{j-1},\theta_j);\ j=1,2,\dots N\}$, where

$$\theta_x = xh; \quad h = \alpha/N \tag{21}$$

and (15) was satisfied at the mid-points $\theta' = \theta_{j-1/2}$ ($j=1,2,\dots N$). Since all of the integrands are continuous, the simplest numerical scheme would be to evaluate the integrals by the mid-point rule. However, the first group of terms, involving I_1, I_2 and I_3, have an apparent singularity at $\theta=\theta'$. We avoided evaluation of the integrand at points $\theta=\theta_{j-1/2}$ by a limiting procedure by estimating the integral over the jth interval as h times the value of the integrand at a point $\theta^*_{j-1/2}$ different from the mid-point $\theta_{j-1/2}$. Because the integrand at least contains a factor $(\alpha-\theta)^{1/2}$ near $\theta=\alpha$, $\theta^*_{j-1/2}$ was chosen so that

$$\int_{\theta_{j-1}}^{\theta_j}(\alpha-\theta)^{1/2}\,\mathrm{d}\theta = h(\alpha-\theta^*_{j-1/2})^{1/2} \tag{22}$$

exactly. Thus,

$$\theta^*_{j-1/2} = \alpha - \frac{4}{9h^2}[(\alpha - (j-1)h)^{3/2} - (\alpha - jh)^{3/2}]^2. \tag{23}$$

This quadrature rule is exact when the integrand has the form $a(\alpha-\theta)^{1/2}+b$ for any constants a and b. It can be checked that $\theta^*_{j-1/2} - \theta_{j-1/2} > 0$, increases monotonically with j, and $\theta^*_{N-1/2} - \theta_{N-1/2} = 0{\cdot}0555\ldots h$. The terms involving $f(\theta')$ and $f'(\theta')$ still require evaluation of these functions at $\theta' = \theta_{j-1/2}$. These are expressed in terms of the unknowns $f(\theta^*_{j-1/2})$ by use of simple extrapolations, based upon approximating $f(\theta)$ in the vicinity of $\theta_{j-1/2}$ as

$$f(\theta) \approx a_j(\alpha-\theta)^{1/2} + b_j, \tag{24}$$

with a_j and b_j fixed by requiring equality at $\theta = \theta^*_{j-1/2}$ and $\theta = \theta^*_{j+1/2}$, with the convention that $\theta^*_{N+1/2} = \alpha$. Thus,

$$f(\theta) \approx \frac{f(\theta^*_{j-1/2})[(\alpha-\theta)^{1/2}-(\alpha-\theta^*_{j+1/2})^{1/2}]+f(\theta^*_{j+1/2})[(\alpha-\theta^*_{j-1/2})^{1/2}-(\alpha-\theta)^{1/2}]}{(\alpha-\theta^*_{j-1/2})^{1/2}-(\alpha-\theta^*_{j+1/2})^{1/2}}, \tag{25}$$

$$f'(\theta) \approx \frac{[f(\theta^*_{j+1/2}) - f(\theta^*_{j-1/2})](\alpha-\theta)^{-1/2}}{2[(\alpha-\theta^*_{j-1/2})^{1/2} - (\alpha-\theta^*_{j+1/2})^{1/2}]}. \tag{26}$$

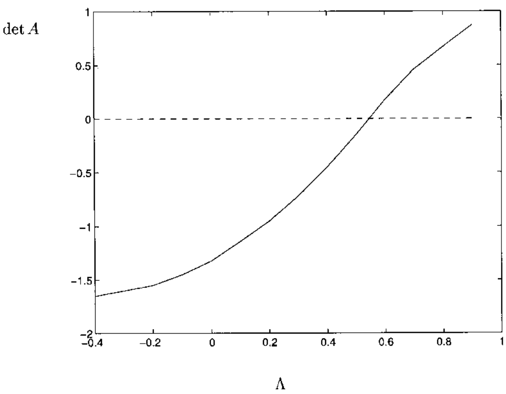

Figure 3. det A as a function of Λ, for $\alpha = \pi/2$, $\nu = 0{\cdot}3$.

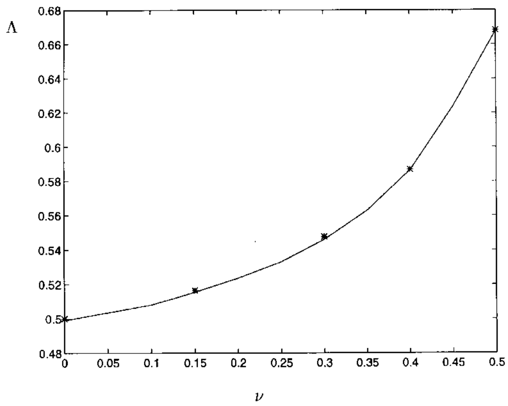

Figure 4. The singularity exponent Λ as a function of the Poisson ratio ν, for $\alpha = \pi/2$. The results obtained by Benthem [1, 2] are marked by $*$.

4. Numerical results

The results of numerical calculations are presented in Figures 3 to 5 which were obtained by taking $N = 100$. In Figure 3 the values of the determinant of A are plotted against Λ. The smallest value of Λ at which the determinant is equal to zero characterizes the singularity in stresses at the corner point. Note that no values of Λ, with $0 > \Lambda > -\frac{1}{2}$, at which det$A = 0$ were found. This means that the displacement field at the corner point is bounded.[3]

[3] This confirms the results of [1, 2] and [5], but disagrees with those of [3, 4].

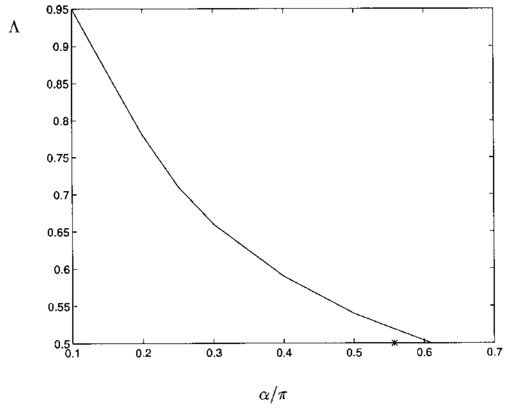

Figure 5. The singularity exponent Λ as a function of the angle α, for $\nu = 0{\cdot}3$. The result of Bažant and Estenssoro [5] is marked by $*$.

Figure 4 exhibits Λ as a function of the Poisson ratio ν. One can see that the values of Λ given by the numerical scheme above agree with those of Benthem [1] shown in Figure 4 by '$*$' for comparison. Note that similar values of Λ have been also obtained by Bažant and Estenssoro [5].[4]

In Figure 5 the exponent Λ is plotted as a function of the angle α of inclination of the crack edge. It decreases from 1 to $\frac{1}{2}$ when α goes from 0^0 to $\sim 110^0$. In Figure 5 we also present the result of Bažant and Estenssoro [5]. The angle α at which $\Lambda = \frac{1}{2}$ (*i.e.* at which the stress singularity at the corner point is the same as at any other point at the crack edge) predicted by the scheme above does not agree with that of [5] but it is much closer to the experimental value $\sim 115^0$ reported by Bell and Feeney [13].

A paper by Glushkov, Glushkova and Lapina [14] appeared after our calculations were completed. Glushkov *et al.* treat a range of problems which involve corner points, including the problem discussed in this article. Their method was, in essence, to apply a Mellin transform to Equation (5) so generating an integral equation in the angular variable (θ in present notation). The corner singularity is analysed by making an assumption equivalent to our Equation (9) Mellin transformed with respect to ρ.

Glushkov *et al.* discretized their equation (relative to present notation) by expressing $f(\theta)$ in the form $f(\theta) = \theta^{\delta_1}(\alpha - \theta)^{\delta_2} g(\theta)$ for some smooth $g(\theta)$. They took $\delta_2 = \frac{1}{2}$ (*c.f.* the present Equation (22)) but claimed a failure to obtain convergence with the choice $\delta_1 = 0$. Instead, their results were computed with $\delta_1 = 1$, with the built-in restriction that $f(0) = 0$, *i.e.* that the opening of the crack at the free surface associated with the most singular term is zero. This is surprising physically and does not conform to what the present numerical scheme generated. The computed values for Λ (actually, Glushkov *et al.* gave $1 - \Lambda$) display the same trends as in Figure 5 but are not in close agreement.

The method described above allows one to evaluate the stress singularity exponent $\Lambda - 1$ at the corner point where the crack edge intersects the boundary. This method can also be applied to a surface-breaking crack propagating dynamically in a plane orthogonal to the free surface. The work is in progress.

[4] Kawai *et al.* [1] obtained $\Lambda = 0{\cdot}3$ for $\nu = 0$ as the lowest root of the equation $\det A = 0$. This disagrees with Benthem [1, 2] and Bažant and Estenssoro [5], as well as with our solution.

Appendix A

Here we give the representations for the quantities I_1 to I_4, J_1 to J_4 and $\tilde{K}_0$ (formulae (10) to (14)). They are as follows

$$
\begin{aligned}
I_1(\theta,\theta') &= \int_0^2 \frac{\tilde{\rho}^{\Lambda+1}\,\mathrm{d}\tilde{\rho}}{(1+\tilde{\rho}^2-2\tilde{\rho}\,\cos(\theta-\theta'))^{3/2}} \\
&= \frac{\Lambda+1}{1-\cos(\theta-\theta')}\int_0^1 \frac{t^\Lambda}{(1-t)^\Lambda}[1-\tfrac{1}{2}t(1+\cos(\theta-\theta'))]^\Lambda\,\mathrm{d}t - I_4(\theta,\theta'),
\end{aligned}
\tag{A1}
$$

$$
\begin{aligned}
I_2(\theta,\theta') &= \frac{1-\Lambda}{\sin^2(\theta-\theta')}\left[1-\frac{1-2\,\cos(\theta-\theta')}{\sqrt{5-4\,\cos(\theta-\theta')}}\right] \\
&\quad + \frac{\Lambda\,\cos^2(\theta-\theta')}{\sin^2(\theta-\theta')}\left(1-\frac{1}{\sqrt{5-4\cos(\theta-\theta')}}\right) \\
&\quad + \frac{2\Lambda\cos(\theta-\theta')\cos[2(\theta-\theta')]}{\sin^2(\theta-\theta')\sqrt{5-4\,\cos(\theta-\theta')}} \\
&\quad + \Lambda\cos(\theta-\theta')\ln\left[\frac{\sqrt{5-4\,\cos(\theta-\theta')}+2-\cos(\theta-\theta')}{1-\cos(\theta-\theta')}\right],
\end{aligned}
\tag{A2}
$$

$$
\begin{aligned}
I_3(\theta,\theta') &= \frac{\cos(\theta-\theta'}{\sin(\theta-\theta')}\left(1-\frac{1}{\sqrt{5-4\,\cos(\theta-\theta')}}\right) \\
&\quad + \frac{2\cos[2(\theta-\theta')]}{\sin(\theta-\theta')\sqrt{5-4\cos(\theta-\theta')}} \\
&\quad + \sin(\theta-\theta')\ln\left[\frac{\sqrt{5-4\cos(\theta-\theta')}+2-\cos(\theta-\theta')}{1-\cos(\theta-\theta')}\right],
\end{aligned}
\tag{A3}
$$

$$
\begin{aligned}
I_4(\theta,\theta') &= \int_2^\infty \frac{\tilde{\rho}^{\Lambda+1}\,\mathrm{d}\tilde{\rho}}{(1+\tilde{\rho}^2-2\tilde{\rho}\,\cos\,(\theta-\theta'))^{3/2}} \\
&= \int_0^{1/3} \frac{(1-t)^{\Lambda+1}\,\mathrm{d}t}{t^\Lambda[1-2t(1-t)(1+\cos(\theta-\theta'))]^{3/2}},
\end{aligned}
\tag{A4}
$$

$$
\begin{aligned}
J_1(\theta,\theta') &= -\frac{1}{\sin(\theta-\theta')}\left[\frac{2-\cos(\theta-\theta')}{\sqrt{5-4\,\cos(\theta-\theta')}}+\cos(\theta-\theta')\right] \\
&\quad + \Lambda\,\sin(\theta-\theta')\ln\left[\frac{\sqrt{5-4\cos(\theta-\theta')}+2-\cos(\theta-\theta')}{1-\cos(\theta-\theta')}\right] \\
&\quad + \frac{2\Lambda\sin(\theta-\theta')}{\sqrt{5-4\cos(\theta-\theta')}},
\end{aligned}
\tag{A5}
$$

$$J_2(\theta,\theta') = \frac{1-2\cos(\theta-\theta')}{\sqrt{5-4\cos(\theta-\theta')}} - 1$$
$$-\cos(\theta-\theta')\ln\left[\frac{\sqrt{5-4\cos(\theta-\theta')}+2-\cos(\theta-\theta')}{1-\cos(\theta-\theta')}\right], \tag{A6}$$

$$J_3(\theta') = \frac{\Lambda}{2}\left[\Pi\left(\delta_{\theta'},\frac{8}{9},\frac{2\sqrt{2}}{3}\right)+\Pi\left(\delta_{\alpha-\theta'},\frac{8}{9},\frac{2\sqrt{2}}{3}\right)\right]$$
$$-(1+\tfrac{3}{2}\Lambda)\left[E\left(\delta_{\theta'},\frac{2\sqrt{2}}{3}\right)+E\left(\delta_{\alpha-\theta'},\frac{2\sqrt{2}}{3}\right)\right]$$
$$-\tfrac{1}{3}(1+5\Lambda)\left[F\left(\delta_{\theta'},\frac{2\sqrt{2}}{3}\right)+F\left(\delta_{\alpha-\theta'},\frac{2\sqrt{2}}{3}\right)\right], \tag{A7}$$

$$J_4(\theta') = \frac{3}{2}\left(\frac{1}{\sqrt{5-4\cos(\alpha-\theta')}}-\frac{1}{\sqrt{5-4\cos\theta'}}\right)$$
$$-\tfrac{3}{2}(\sqrt{5-4\cos(\alpha-\theta')}-\sqrt{5-4\cos\theta'}). \tag{A8}$$

In (A7) E, F and Π are the elliptic integrals, $\delta_\theta = \sin^{-1}\{3\sqrt{(1-\cos\theta)/[2(5-4\cos\theta)]}\}$. $\tilde{K}_0(\Lambda;\theta,\theta')$ is the Mellin transform of $K_0(\rho,\theta;\rho',\theta') = (\rho')^{-3}K_0(\tilde{\rho};\theta,\theta')$ with respect to $\tilde{\rho}=\rho/\rho'$, with $s=\Lambda+2$,

$$\tilde{K}_0(\Lambda;\theta,\theta')$$
$$= -\frac{E}{8(1-\nu^2)}\frac{\Lambda(\Lambda+1)}{\sin\pi\Lambda}\left\{(1-2\kappa)\frac{P_\Lambda^{-1}(-\cos(\theta+\theta'))}{\sqrt{1-\cos^2(\theta+\theta')}}\right.$$
$$+3(\Lambda+2)(1-\Lambda)\left[1-\frac{1-\cos(\theta-\theta')}{1-\cos(\theta+\theta')}\right]\frac{P_\Lambda^{-2}(-\cos(\theta+\theta'))}{1+\cos(\theta+\theta')}$$
$$+4\nu(1-2\nu)\left\{[(\Lambda+2)\cos^2\theta+(1-\Lambda)\cos^2\theta']\frac{P_\Lambda^{-1}(-\cos(\theta+\theta'))}{\sqrt{1-\cos^2(\theta+\theta')}}\right.$$
$$+(\Lambda+2)(1-\Lambda)\left[-\frac{\cos(\theta+\theta')(\sin^2\theta+\sin^2\theta')}{1-\cos(\theta+\theta')}+\frac{\cos(\theta-\theta')-\cos(\theta+\theta')}{1-\cos(\theta+\theta')}\right]$$
$$\left.\times\frac{P_\Lambda^{-2}(-\cos(\theta+\theta'))}{1+\cos(\theta+\theta')}\right\}+\frac{\sin\pi\Lambda}{\pi\Lambda(\Lambda+1)}$$
$$\times\int_0^\infty\frac{12(1-\nu)(1-2\nu)\tilde{\rho}^{\Lambda+1}\,\mathrm{d}\tilde{\rho}}{\sqrt{1+\tilde{\rho}^2-2\tilde{\rho}\cos(\theta+\theta')}}$$
$$\left.\frac{}{[\sqrt{1+\tilde{\rho}^2-2\tilde{\rho}\cos(\theta+\theta')}+\tilde{\rho}\,\sin\theta+\sin\theta']^2}\right\}. \tag{A9}$$

In (A9) P_Λ^{-1}, P_Λ^{-2} are the associated Legendre functions, $\kappa = 3 - 4\nu$. By the coordinate transformation $\tilde{\rho} = (1-t)/t$ the last integral in (A9) is reduced to an integral over a finite interval,

$$\int_0^\infty \frac{\tilde{\rho}^{\Lambda+1}\,\mathrm{d}\tilde{\rho}}{\sqrt{1+\tilde{\rho}^2-2\tilde{\rho}\,\cos(\theta+\theta')}[\sqrt{1+\tilde{\rho}^2-2\tilde{\rho}\,\cos(\theta+\theta')}+\tilde{\rho}\,\sin\theta+\sin\theta']^2}$$

$$= \int_0^1 \frac{(1-t)^{\Lambda+1}\,\mathrm{d}t}{t^\Lambda\sqrt{1-2t(1-t)(1+\cos(\theta+\theta'))}[\sqrt{1-2t(1-t)(1+\cos(\theta+\theta'))}+t(\sin\theta'-\sin\theta)+\sin\theta]^2},$$

and it is evaluated numerically. The same coordinate transformation was used in (A4).

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